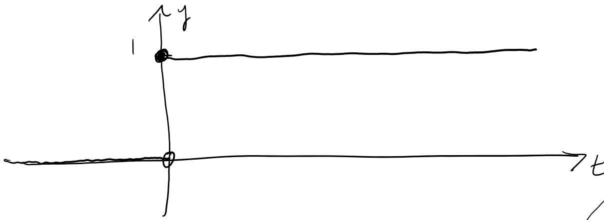
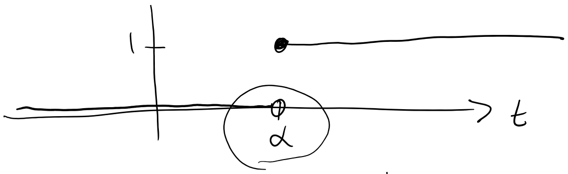
Section 5.2 Laplace Transform Pairs

Wednesday, April 15, 2020 9:35 AM

Heaviside step function



$$h(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$h(t-\alpha) = \begin{cases} 1 & t \geq \alpha \\ 0 & t < \alpha \end{cases}$$

$$\mathcal{L}\{h(t)\} = \int_0^\infty e^{-st}(1)\, dt = \boxed{\frac{1}{s} \quad s > 0}$$

$$\mathcal{L}\{h(t-\alpha)\} = \int_0^\infty e^{-st}\, h(t-\alpha)\, dt$$

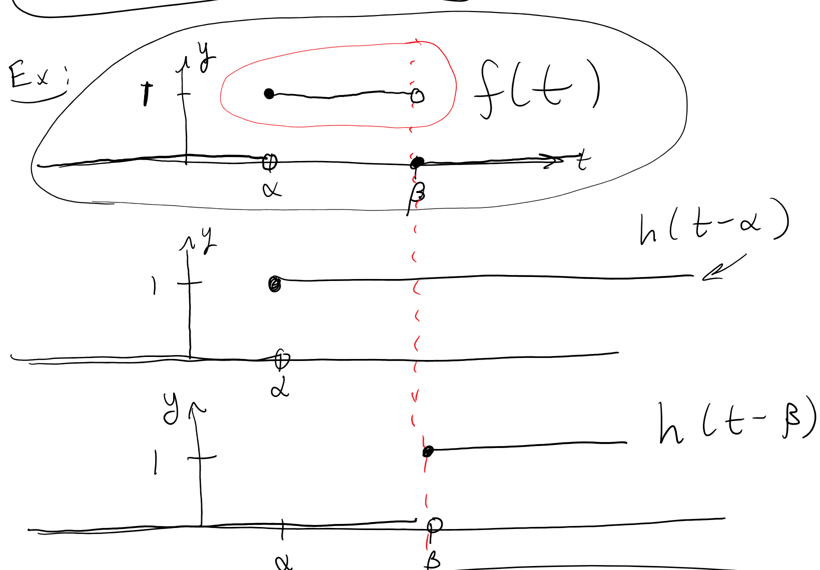
$$= -\frac{1}{s} \int e^{-st}(-s\, dt) \qquad u = -st, \quad du = -s\,dt$$

$$= -\frac{1}{s}\left[e^{-st}\right]_\alpha^\infty$$

$$= -\frac{1}{s}\left[0 - e^{-\alpha s}\right]$$

$$\boxed{\mathcal{L}\{h(t-\alpha)\} = \frac{e^{-\alpha s}}{s}} \qquad s > \alpha$$

Ex:



$$\boxed{f(t) = h(t-\alpha) - h(t-\beta)}$$

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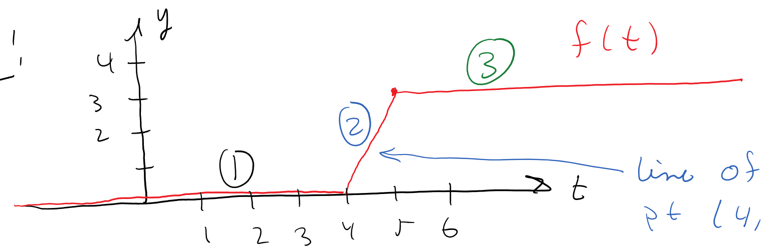
$$\mathcal{L}\{f(t)\} = \mathcal{L}\{h(t-\alpha)\} - \mathcal{L}\{h(t-\beta)\}$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{e^{-\alpha s}}{s} - \frac{e^{-\beta s}}{s}} \qquad s > \beta$$

Ex!



line of slope 3
pt (4,0)

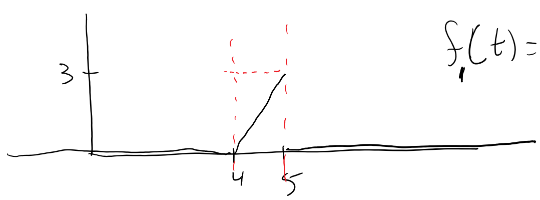
$y - y_1 = m(t - t_1)$

$y = 3(t-4)$

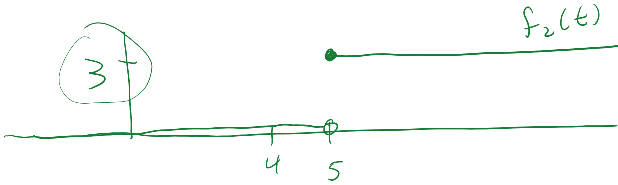
$$f(t) = \begin{cases} 0 & t < 4 \\ 3(t-4) & 4 \le t \le 5 \\ 3 & t > 5 \end{cases}$$

1 between 4 & 5

$$f_1(t) = 3(t-4)\underbrace{\left(h(t-4) - h(t-5)\right)}$$



$$f_2(t) = 3\left(h(t-5)\right)$$



$$f(t) = f_1(t) + f_2(t)$$

$$= 3(t-4)\left(h(t-4) - h(t-5)\right) + 3h(t-5)$$

$$= 3(t-4)\,h(t-4) - 3(t-4)\,h(t-5) + 3h(t-5)$$

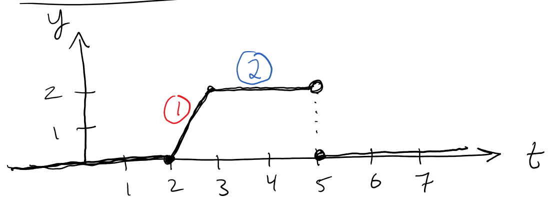
$$= 3(t-4)\,h(t-4) + h(t-5)\left[-3t + 12 + 3\right]$$

$$= 3(t-4)\,h(t-4) + h(t-5)\left[-3(t-5)\right]$$

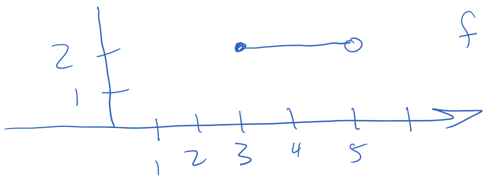
$$\boxed{= 3(t-4)\,h(t-4) - 3(t-5)\,h(t-5)}$$

$$\boxed{= 3(t-4)h(t-4) - 3(t-5)h(t-5)}$$

Ex:



$$f_1(t) = 2(t-2)\left[h(t-2) - h(t-3)\right]$$

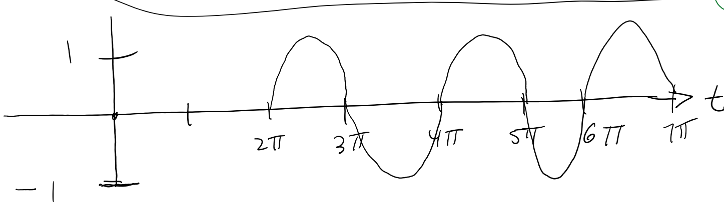


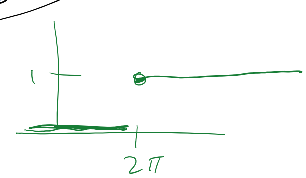
$$f_2(t) = 2\left[h(t-3) - h(t-5)\right]$$

$$\begin{aligned}
f(t) &= f_1(t) + f_2(t) \\
&= 2(t-2)\left[h(t-2) - h(t-3)\right] + 2\left[h(t-3) - h(t-5)\right] \\
&= 2(t-2)h(t-2) - 2(t-2)h(t-3) + 2h(t-3) - 2h(t-5) \\
&= 2(t-2)h(t-2) + h(t-3)\left[\underbrace{2 - 2t + 4}_{-2t+6}\right] - 2h(t-5)
\end{aligned}$$

$$\boxed{= 2(t-2)h(t-2) + \left[-2(t-3)\right]h(t-3) - 2h(t-5)}$$

Ex: graph $f(t) = \sin(t-2\pi)\, h(t-2\pi)$





OR $g(t) = \sin(t)\, h(t-2\pi)$

5.2.1 Shift Theorems

$\mathcal{L}\{f(t)\} = F(s)$

1. $\mathcal{L}\{e^{\alpha t} f(t)\} = F(s-\alpha)$

   where $\mathcal{L}\{f(t)\} = F(t)$ See #9 on table 1

2. $\mathcal{L}\{f(t-\alpha)h(t-\alpha)\} = e^{-\alpha s}F(s)$

Laplace shift from time

Ex: $\mathcal{L}\{\sin(t-2\pi)\, h(t-2\pi)\} = \boxed{e^{-2\pi s}\left(\frac{1}{s^2+1}\right)}$

$f(t-\alpha) = \sin(t-2\pi)$

5. $\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
6. $\cos \omega t$	$\frac{s}{s^2+\omega^2}$

$f(t) = \sin t \qquad \alpha = 2\pi$

$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1^2}$ #5

Ex: $\mathcal{L}\{e^{2t}\cos(3t)\} = F(s-2) = \boxed{\frac{s-2}{(s-2)^2+9}}$

$\alpha = 2$

$f(t) = \cos(3t)$

#6 $\mathcal{L}\{f(t)\} = \frac{s}{s^2+3^2} = F(s)$

Ex: $\mathcal{L}\{3t^2+2t+1\} = \boxed{3\left(\frac{2!}{s^3}\right) + 2\left(\frac{1!}{s^2}\right) + \frac{1}{s}}$

3. t^n,	$n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

Ex: $\mathcal{L}\{e^{4t}(3t^2+2t+1)\} = F(s-4)$

$f(t) = 3t^2+2t+1$

$\alpha = 4$

$$\boxed{= 3\left(\frac{2}{(s-4)^3}\right) + 2\left(\frac{1}{(s-4)^2}\right) + \frac{1}{s-4}}$$

Ex: $\mathcal{L}\{e^{3t-3}h(t-1)\} = \mathcal{L}\{e^{3(t-1)}h(t-1)\}$

$\mathcal{L}\{f(t-\alpha)h(t-\alpha)\} = e^{-\alpha s}F(s)$

$$\mathcal{L}\{f(t-\alpha)\,h(t-\alpha)\} = e^{-\alpha s} F(s)$$

$$f(t) = e^{3t}$$

$$\alpha = 1$$

$$f(t-1) = e^{3(t-1)}$$

$$\mathcal{L}\{e^{\alpha t}\} = \frac{1}{s-\alpha} \quad \text{row \#4}$$

$$\mathcal{L}\{f(t)\} = F(s) = \frac{1}{s-3}$$

$$\boxed{\mathcal{L}\{e^{3(t-1)}h(t-1)\} = e^{-s}\left(\frac{1}{s-3}\right)}$$

Ex: Find $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s} + \frac{24}{s^2}\right\} = \boxed{3 + 24t}$

$$\mathcal{L}^{-1} \quad s^{n+1} = t^n$$

$$\mathcal{L}^{-1}\left\{\frac{3}{s}\right\} = 3\,\mathcal{L}^{-1}\left\{\frac{1}{s^1}\right\} = 3t^0 = 3$$

$n=0$

$$24\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = 24t^1$$

$n=1$

Ex: $F(s) = \dfrac{2s-4}{(s-2)^2+9} = 2\left(\dfrac{s-2}{(s-2)^2+9}\right)$

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
12. $e^{\alpha t}\sin\omega t$	$\dfrac{\omega}{(s-\alpha)^2+\omega^2}$
13. $e^{\alpha t}\cos\omega t$	$\dfrac{s-\alpha}{(s-\alpha)^2+\omega^2}$

$\alpha = 2 \qquad \omega = 3$

$$\mathcal{L}^{-1}\left\{2\left(\frac{s-2}{(s-2)^2+9}\right)\right\} = \boxed{2e^{2t}\cos 3t}$$

Ex: $G(s) = e^{-2s}\dfrac{3}{s^2+9} \qquad \alpha = 2$

shift thm #2

$$\mathcal{L}^{-1}\{e^{-\alpha s}F(s)\} = f(t-\alpha)\,h(t-\alpha) \quad \text{row \#14 on table}$$

$$\mathcal{L}^{-1}\left(\frac{\omega}{s^2+\omega^2}\right\} = \sin\omega t$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$f(t) = \sin(3t)$$

$$f(t) = \sin(3t)$$

$$\mathcal{L}^{-1}\left\{ e^{-2s}\left(\frac{s}{s^2+9}\right)\right\} = \boxed{\sin(3(t-2))\,\underbrace{h(t-2)}}$$

Software $\rightarrow$ $\begin{cases} \sin(3(t-2)) & t \geq 2 \\ 0 & t < 2 \end{cases}$

Ex: $G(s) = \dfrac{4s-6}{s^2-2s+10}$ $\leftarrow$ complete the square.

Need $(s-1)$ in top

$$\frac{4s-6}{s^2-2s+1+10-1} = \frac{4s-6}{(s-1)^2+3^2} = \frac{4s-4-2}{(s-1)^2+3^2}$$

$$= \frac{4(s-1)-2}{(s-1)^2+3^2} = \frac{4(s-1)}{(s-1)^2+3^2} - \frac{2}{(s-1)^2+3^2}$$

12. $e^{\alpha t}\sin\omega t$	$\dfrac{\omega}{(s-\alpha)^2+\omega^2}$
13. $e^{\alpha t}\cos\omega t$	$\dfrac{s-\alpha}{(s-\alpha)^2+\omega^2}$

$\alpha = 1$

$$4\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+3^2}\right\} - \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s-1)^2+3^2}\right\}$$

$$\boxed{= 4e^{t}\cos(3t) - \frac{2}{3}e^{t}\sin(3t)}$$

5.2.3 Laplace transforms of derivatives

number

1. $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0)$

2. $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

3. $\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \dfrac{\mathcal{L}\{f(t)\}}{s} = \dfrac{F(s)}{s}$

Ex: $\mathcal{L}\left\{\dfrac{dy}{dt} + 6y(t) + 9\int_0^t y(\tau)\,d\tau = 1\right\}$, $y(0) = 0$

$\mathcal{L}\{y'\} + 6\mathcal{L}\{y(t)\} + 9\mathcal{L}\left\{\int_0^t y(\tau)\,d\tau\right\} = \mathcal{L}\{1\}$

$sY(s) - \cancel{y(0)}^{\,0} + 6Y(s) + 9\dfrac{Y(s)}{s} = \dfrac{1}{s}$

$Y(s)\left(s + 6 + \dfrac{9}{s}\right) = \dfrac{1}{s}$

$Y(s) = \dfrac{1}{s\left(s + 6 + \frac{9}{s}\right)}$

$Y(s) = \dfrac{1}{s^2 + 6s + 9} = \dfrac{1}{(s+3)^2}$

row 11 $\quad \mathcal{L}^{-1}\{Y(s)\} = \boxed{y(t) = e^{-3t}\,t}$

$n = 1$

$\alpha = -3$

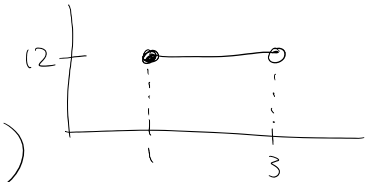
$\mathcal{L}^{-1}\left\{\dfrac{n!}{(s-\alpha)^{n+1}}\right\} = e^{\alpha t}\,t^n$

Ex: $y' + 4y = g(t)$ $\qquad y(0) = 0$

Initial value problem.

$g(t) = \begin{cases} 0 & 0 \le t < 1 \\ 12 & 1 \le t < 3 \end{cases}$

$$g(t)=\begin{cases}0 & 0\le t<1\\ 12 & 1\le t<3\\ 0 & 3\le t\end{cases}$$



$$g(t)=12\left(h(t-1)-h(t-3)\right)$$

$$\mathcal{L}\{y'\}+\mathcal{L}\{4y\}=\mathcal{L}\{12\,(h(t-1))\}-\mathcal{L}\{12\,h(t-3)\}$$

$$sY(s)-\cancel{y(0)}^{\,0}+4Y(s)=12\,\frac{e^{-s}}{s}-12\,\frac{e^{-3s}}{s}$$

$$Y(s)(s+4)=12\left(\frac{e^{-s}-e^{-3s}}{s}\right)$$

$$Y(s)=\frac{12}{s(s+4)}\left(e^{-s}-e^{-3s}\right)$$

Partial fractions.
section 5.3

$$\frac{12}{s(s+4)}=\frac{\cancel{A}\ 3}{s}+\frac{\cancel{B}\ -3}{s+4}$$

$$Y(s)=\left(\frac{3}{s}-\frac{3}{s+4}\right)\left(e^{-s}-e^{-3s}\right)$$

1. $h(t)=\begin{cases}1 & t\ge 0\\ 0, & t<0\end{cases}$	$\frac{1}{s}$
2. 1	$\frac{1}{s}$
3. t^n, $\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$
4. e^{at}	$\frac{1}{s-a}$
14. $f(t-\alpha)h(t-\alpha)$, $\quad(\alpha\ge 0)$, with $\lvert f(t)\rvert\le Me^{at}$	$e^{-\alpha s}F(s)$

$F(s)$

$$Y(s)=\left(\frac{3}{s}-\frac{3}{s+4}\right)e^{-s}-\left(\frac{3}{s}-\frac{3}{s+4}\right)e^{-3s}$$

$$Y(s) = \underbrace{\left(\frac{3}{s} - \frac{3}{s+4}\right)}_{F(s)} e^{-s} - \underbrace{\left(\frac{3}{s} - \frac{3}{s+4}\right)}_{F(s)} e^{-3s}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left(\frac{3}{s} - \frac{3}{s+4}\right)$$

$\alpha = -4$

$$f(t) = 3 - 3e^{-4t}$$

$$f(t-\alpha) = 3 - 3e^{-4(t-\alpha)}$$

$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\left(\frac{3}{s} - \frac{3}{s+4}\right)e^{-s}\right\} - \mathcal{L}^{-1}\left\{\left(\frac{3}{s} - \frac{3}{s+4}\right)e^{-3s}\right\}$$

$$\boxed{y(t) = \left(3 - 3e^{-4(t-1)}\right)h(t-1) - \left(3 - 3e^{-4(t-3)}\right)h(t-3)}$$